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Strategic Patenting and the Tragedy of Anticommons: A Closer Look at Firms'™ Patenting Behavior

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Abstract

Prior research postulates an inconclusive causal effect of fragmentation of property rights on firms' patenting behavior. On one hand, studies that emanate from concepts such as 'patent thickets' and

Strategic Patenting and the Tragedy of Anticommons: A Closer Look at Firms' Patenting Behavior

Abstract: Prior research postulates an inconclusive causal effect of fragmentation of property rights on firms' patenting behavior. On one hand, studies that emanate from concepts such as "patent thickets" and "the tragedy of anticommons" suggest that a dense overlapping set of fragmented property rights can result in technological stagnation and, hence, in a decline in firms' patenting rate. On the other hand, research derived from such notions as "strategic patenting" argues that firms facing a fragmented external technology market will engage in patent proliferation strategies. In this paper, I develop an integrated theoretical model that captures the main features of both arguments. Indeed, the model posits that firms are more likely to engage in patent proliferation strategies when their technologies draw on patents assigned to a disperse set of outside entities. However, as the overlapping claims of these external right holders increase, the expected infringement costs associated with a patent proliferation strategy may exceed its benefits. The model is empirically tested using panel data of patenting behavior of public firms in the semiconductors industry from 1980 to 1999. The empirical results support the model predictions.

1. Introduction

The past decade or so has witnessed a new wave of studies focusing on the effects of property rights fragmentation on firms' patenting behavior. Two seemingly inconsistent arguments have emerged as a result. On the one hand, studies that emanate from concepts such as "patent thickets" (Shapiro, 2001; Lemley and Shapiro, 2007), "the tragedy of anticommons" (Heller and Eisenberg, 1998), and "patent holdup" (Lemley and Shapiro, 2005) suggest that a dense web of overlapping patents owned by too many individual players in a technology area leads to subsequent underinvestment in cumulative innovation and a decline in the patenting rate of firms in that area. On the other hand, research that is derived from such notions as "strategic patenting" (Levin, Klevorick, Nelson, & Winter, 1987; Bessen and Hunt, 2007), and "patent proliferation" (Hall and Ziedonis, 2001; Ziedonis, 2004; Lemley and Shapiro, 2005) posit that when firms face

a fragmented external technology market on which they have to build their new technologies, they tend to pursue aggressive patenting behavior and increase their patent portfolio size.

Considering that both classes of arguments are supported theoretically and empirically in the literature, one may wonder under what conditions each of these arguments provides a better prediction of the firms' patenting behavior. This paper is the first attempt to integrate these two arguments into one theoretical framework and investigate their joint effect on firms' patenting behavior.

To do so, I first develop a theoretical model which captures the main features of both arguments. Four main predictions arise from the theoretical model. Firstly, firms will patent more aggressively when their new technologies can potentially infringe on several patents assigned to a disperse set of patent holders. This is in line with the "strategic patenting" argument. Secondly, the extent to which these patent holders claim overlapping ideas and technologies negatively impacts firms' patenting rate. It is consistent with the "tragedy of anticommons" and other arguments in the first category. Furthermore, the model suggests that the absolute impact of overlapping external property rights is relatively stronger than that of disperse external patent rights. This means that when firms' external technology market is both highly fragmented and also overcrowded by overlapping patent rights assigned to different entities, the expected costs of engaging in a patent proliferation strategy can surpass its expected benefits. Finally, the model predicts that firms whose previous patents are extensively used by other players in the technology market are less likely to engage in patent proliferation strategies.

In the empirical section of the paper, I test the validity of these predictions using data on firms' patenting behavior in semiconductors industry between 1980 and 1999. Based on the idea underlying Ziedonis' (2004) "fragmentation index", I develop two new measures, "overlap

density index” and “portfolio strength index” to measure the extent to which external patent holders claim overlapping regions of technology space, and the strength of the firms’ initial patent portfolio, respectively. Using “fragmentation index” along with these two new measures, I find empirical support for the predictions derived from the model.

This study extends the current academic literature by providing a more holistic image of how the external allocation of property rights can affect the firms’ patenting behavior. It also provides an integrated theoretical framework which can be used as basis in future theoretical and empirical studies in the field. From a policy perspective, this paper, along with several other studies in the field (ex: Bessen and Meurer, 2008; Burk and Lemley, 2009), calls for fundamental restructuring of the current patent system. In particular, it illustrates how a dense web of fragmented and overlapping property rights may eventually lead to technological stagnation especially in areas where innovations are cumulative and complementary.

The structure of the paper is as follows. In the next section I will briefly overview the ideas underlying each of the aforementioned arguments. Following that, using the main features of both arguments, I develop a simple model of firms’ patenting behavior. In the subsequent section, I test the predictions of the model in the semiconductor industry. Finally, I discuss the findings and their implications and conclude with limitations and avenues for future research.

2. The Tragedy of Anticommons vs. Strategic Patenting

The “tragedy of anticommons” refers to the situation where the existence of multiple gatekeepers for a common resource can lead to an underutilization of that resource (Heller, 1998). Heller and Eisenberg (1998) point out the case of biotechnology patents and argue that when firms need to negotiate with large numbers of patent holders, they may face excessive transaction costs which

can potentially diminish their incentives to invest in relevant research. While the early versions of the argument were focused primarily on the excessive transaction costs associated with numerous ex ante negotiations, the more recent accounts of the idea, mainly under the notions of “patent holdup” and “patent thickets”, also highlight the excessive costs associated with over-valued licensing royalty rates and substantial expected ex post infringement costs (Shapiro, 2001; Merges, 2001; Lemley and Shapiro, 2005).

Previous research suggests that the current patent system is in fact creating a crowded “thicket” of overlapping intellectual property rights through which an organization needs to navigate in order to develop and commercialize its new technologies (Lemley and Shapiro, 2007).

Innovation will be stifled as a result. Using bargaining theory, Lemley and Shapiro (2007) show that the threat to obtain a permanent injunction can greatly increase the negotiation power of the patent holders which can result in excessive royalty rates and, hence, holdup problems. They also show that when multiple patents by different firms read on a single product, the holdup problems will magnify; a situation they call “royalty stacking”. All these studies rely on the assumption that when a technological area is overcrowded and several right holders have overlapping claims, any further technological development in the field bears the risk of infringing upon several previous patents, and, hence, may introduce considerable infringement costs. Therefore, the bottom line is that fragmented and overlapping property rights can hold back cumulative innovations and further technological developments.

A few empirical studies support these arguments to some extent. Lerner (1995) shows that firms with high litigation costs are indeed more likely to avoid highly fragmented technology classes with too many awards granted to other firms, particularly to firms with low litigation costs .

Based on extensive survey results, Razgaitis (2004, 2005) reported that more than 40% of

licensing negotiations among organizations with considerable licensing activities were terminated unsuccessfully. Failure was reported to occur because there were either too many parties involved in the negotiation or because a useful bundle of IP could not be assembled. Furthermore, focusing on third-generation cellular telephones and Wi-Fi systems, Lemley and Shapiro (2007) show that “royalty stacking” can in fact cause serious hold-up problems particularly in the standard-setting contexts where numerous patents can read on a single product standard.

On the opposite end of the spectrum, studies under the notion of “strategic patenting” predict that when firms face a fragmented external technology market they will pursue aggressive patenting behavior to increase their portfolio size and, as a result, their bargaining power in ex post infringement disputes or licensing negotiations (Cohen, Nelson, and Walsh, 2000; Hall and Ziedonis, 2001; Ziedonis, 2004). Bessen (2003) and Hunt (2006) demonstrate that, in principle, whenever a new technology or product reads on many external patents and the cost of patenting is sufficiently low, firms will engage in aggressive patenting behavior, i.e. “strategic patenting”.

There is relatively more empirical support for the “strategic patenting” argument in the literature. Based on the results from an extensive survey of R&D managers in different industries, Cohen et al. (2000) report that, especially in complex industries, the two main motives for extensive patenting, besides prevention of copying, are that patents are useful negotiation tools and they also help prevent lawsuits. Studying the firms in Semiconductors, Hall and Ziedonis (2001) and later Ziedonis (2004) show that the strengthening of U.S. patent rights during 1980s has created “patent amassing” and “patent portfolio races” among capital-intensive firms aimed at reducing the risk of being “fenced in” by external patent holders and achieving a stronger position in licensing negotiations with them. Similarly, Bessen and Hunt (2007) find evidence of strategic

Strategic Patenting and the Tragedy of Anticommons

patenting in software patents. In brief, these studies suggest that facing a fragmented technology market, firms will patent heavily to gain a better bargaining position in potential infringement disputes and future negotiations.

In fact, both factions emphasize the same features of the current patenting system as the fundamental driving force of their predictions: patents' boundaries are blurry and difficult to demarcate (Lemley and Shapiro, 2005); firms can claim overlapping pieces of complementary technologies (Shapiro, 2001); and technological development happens in a cumulative and sequential manner (Bessen and Maskin, 2009). However, while the former puts its emphasis on significant infringement costs, royalty stacking and excessive transaction costs, the latter underscores the importance of large portfolio size in ex post infringement disputes and licensing negotiations.

In order to integrate the ideas behind these two arguments, in the next section, I develop a theoretical model which captures the main features of both. Using the ideas developed mainly in Hall and Ziedonis (2001), Ziedonis (2004) and Bessen (2003), I start with a simple model in which a larger patent portfolio increases the expected chance of winning of a firm in potential infringement disputes. Then, I extend the model to include the expected costs which may arise as the result of new patents potentially infringing on some other patents in overcrowded technological areas. Finally, I extend the model to include the royalty and transaction costs associated with ex ante licensing negotiations. With the semiconductor industry in mind, then I derive four empirically testable predictions from the model.

3. An Integrated Model of Patenting Behavior

Consider a hypothetical firm deciding to invest in a new technology knowing that it might infringe on M patents held by k different firms. Without loss of generality, assume that all these patent holders have the same number of patents ($m = M/K$) with the same quality. In principle, the firm has two options to protect its new technology: negotiating with each of the patent holders ex ante and securing a license (or an alternative contractual agreement) to use their patents, or proceeding without permission from patent holders and relying on ex post defensive actions in case any of the patent owners file an infringement lawsuit to obtain an injunction.

For now, I assume that the focal firm has proceeded with no ex ante negotiations and relies only on defending itself ex post by increasing its relevant patent portfolio size¹. This situation can happen due to numerous reasons. For example, the firm may find it more profitable to defend its technology ex post rather than paying a high stacked royalty rate to several patentees. It can also happen when the firm designed its technology to include a feature for which a patent has subsequently issued by another firm. Or the firm may initially be unaware of the existence of such patents and just later it learns about them (Lemley and Shapiro, 2007). Also, it may be the case that the patent holder strategically files its patent with some delay to put itself into a better bargaining position once the focal firm has already made its investments (Lemley and Shapiro, 2007). Anand and Khanna (2000) show that in fact only 5% or 6% of licensing agreements occur ex ante in computers and electronics industries. Nevertheless, later I relax this assumption and show that the results generally hold when the firm uses a mixture of both ex ante and ex post strategies.

¹Note that in this study we only focus on the defensive use of patents. Nevertheless, previous studies show that firms may also patent to block competitors from patenting in related technologies especially in discrete technologies such as biotechnology (Cohen et al., 2000). Lemley (2000) and Hall, Jaffe, and Trajtenberg (2005) point out that patents can also be used to obtain financing and boost market valuation. Lemley and Shapiro (2005) also point out the role of patents as lottery tickets: firms may patent extensively in several related technologies with the hope that a few of their patents may pay off greatly. However, here we only consider the defensive aspect of patenting: firms patent to protect their technologies from copying and themselves from being fenced in or sued by their competitors.

3.1. The Base Model

All different explanations of defensive patent proliferation mainly rely on the premise that a stronger relevant patent portfolio will increase the chance of the firm to win in potential infringement lawsuits (Cohen et al., 2000; Hall and Ziedonis, 2001; Bessen, 2003)². To implement this feature in the model, assume that the probability that a firm wins in an infringement dispute against another firm is proportional to their relative portfolio size:

$$\Pr(win) = \frac{n}{n + m} \quad (1)$$

where $\Pr(win)$ represents the probability that a firm with n relevant patents wins in an infringement dispute against another firm with a portfolio of m patents. To account for the differences in the quality (i.e. enforceability degree) of different patents, m can be normalized based on the average quality of the patents in the latter's portfolio. For example if both firms have one patent and the quality of the first firm's patent is half that of the other's, the latter's number of patents can be multiplied by two to account for the higher quality of its patent and lower chance of the former's winning in any potential infringement dispute. Also, for now, assume that the only cost associated with patent proliferation is the cost of filing and prosecuting the patent applications.

Considering that the focal firm's new technology draws on patents by k different patent holders, from an ex ante point of view, the firm faces k potential infringement lawsuits³. Of course, from

²It is important to note that only patents with some relevant features can be considered to be in a single patent portfolio. A firm's patent in the field of aerospace does not add much value in licensing negotiations over some pharmaceutical patents. One exception might be that two firms patent in both pharmaceuticals and aerospace areas so that they can exchange one patent in one area for another in the other area, however these circumstances are far from common to be considered as a regularity in the model.

³ For simplification we assume that no matter how many patents of a firm are infringed by another firm, the former will file only one lawsuit to defend its rights. Relaxing this assumption does not change the predictions of the model.

an ex post point of view, some or all of these infringement lawsuits may never occur. However the huge costs of injunction besides the lucrative benefits of filing a lawsuit against the focal firm once it has already developed its technology implies that the firm should prepare itself to defend in k potential infringement disputes; each can potentially cost it a hypothetical amount of c_i in case it loses the case in the court or agrees to settle the case by paying some kind of licensing fee to the patent holder (which hypothetically should cost the firm equally)⁴. Taking into account the cost and probability of losing different numbers of disputes, the ex ante expected profit of the focal firm will be

$$\begin{aligned}
 E(\pi) &= \pi_m - \sum_{j=1}^k [j \cdot c_i \cdot (1 - \Pr(\text{win}))^j \cdot \Pr(\text{win})^{k-j}] - n \cdot c_p \\
 &= \pi_m - k \cdot c_i \cdot \frac{m}{n + n_0 + m} - n \cdot c_p
 \end{aligned} \tag{2}$$

where π_m is the profit that the firm could make given that there was no threat of infringement and no need for patent proliferation; n_0 stands for the number of relevant initial valid patents in the focal firm's patent portfolio before engaging in the development of the new technology; and n stands for the number of new patents that the firm needs to file to increase the size of its patent portfolio to protect its position and maximize its expected profit. Using the first order condition to find the optimal number of new patents yields:

$$n^* = \sqrt{\frac{k \cdot c_i \cdot m}{c_p}} - m - n_0 \tag{3}$$

⁴ Note that infringement costs, i.e. c_i , may include several costs such as actual damages and any profits made by the infringing party due to the use of the copyrighted work, the cost of redesigning the technology, etc.

As it can be seen, the number of new patents required is increasing in the number of patent holders (k), and the infringement cost (c_i). It is also decreasing in the cost of filing and prosecuting new patents (c_p) and the initial strength of the relevant patent portfolio (n_0).

The predicted positive relation between the fragmentation level (here the number of patent holders, i.e. k) and the number of new patents required is in line with the “strategic patenting” argument. The model shows that as the number of the potential infringement disputes grows, the expected cost associated with these infringements also increases. As a result, the firm needs a larger (i.e. stronger) patent portfolio to decrease the chance of losing in future potential disputes.

3.2. The Effect of Overlap Density

The model developed in the previous section relies on the very strong assumption that the only cost associated with the set of new patents is the cost of filing and prosecuting them. However, in a more realistic situation, new patents may indeed infringe on some other patents previously filed by other players in the technology market. In other words, an important requirement for the effectiveness of the patent proliferation strategy is that new patents do not put the firm in a more fragile situation by increasing the expected cost of infringement furthermore. However, in a densely crowded technology market where any new patent might infringe on a number of previously issued patents, the potential costs of infringement might exceed the benefits of a stronger patent portfolio. The extra cost of new patents actually represents the idea behind the “tragedy of anticommons”: the new patents’ costs (licensing, transaction, or infringement costs) might go beyond their benefits. In this section, I first add the potential infringement costs of new

patents. Once I add the licensing and transaction costs to the model in the next section, I will also consider the possibility of extra licenses for new patents⁵.

To implement the potential cost of infringement by new patents to the previous model, I assume that a set of n new patents can potentially infringe on some other patents owned by $n \times d$ firms where d (overlap density) represents the crowdedness of the technological space and can get any real value equal to or larger than zero. One can think of d as the expected number of patent overlaps (at the firm level) which can occur when a firm puts a new patent in the technological space. Evidently d increases as the total number of patentees and the average number of their patents increase⁶. Updating the expected profit of the firm with new potential infringement costs yields:

$$E(\pi) = \pi_m - (k + nd) \cdot c_i \cdot \frac{m}{n + n_0 + m} - n \cdot c_p \quad (4)$$

Maximizing the expected profit with respect to the portfolio size gives

$$n^* = \sqrt{\frac{(k - d(m + n_0)) \cdot c_i \cdot m}{c_p}} - m - n_0 \quad (5)$$

Similar to the previous condition, the number of new patents required is increasing in the number of patent holders (k), and the infringement cost (c_i). It is also decreasing in the cost of filing and prosecuting new patents (c_p) and the strength of the initial patent portfolio (n_0). In addition, now the number of new patents required is decreasing in the overlap density level, d . Taking the

⁵ Indeed it is unlikely that firms engage in any licensing negotiations for their defensive patents. After all, these patents are partially introduced to prevent excessive ex ante royalties and transaction costs.

⁶ Considering that firms will deliberately put their new patents in the less dense parts of the technological space, it is plausible to imagine that on average the density of the space increases uniformly.

derivatives of n^* with respect to k and d provides us some more insights on the simultaneous effect of fragmentation and density:

$$\frac{\partial n^*}{\partial k} = \frac{1}{2} \sqrt{\frac{c_p}{c_i \cdot m}} \frac{1}{\sqrt{k - d(m + n_0)}} > 0 \quad (6)$$

$$\frac{\partial n^*}{\partial d} = \frac{-(m + n_0)}{2} \sqrt{\frac{c_p}{c_i \cdot m}} \frac{1}{\sqrt{k - d(m + n_0)}} < 0 \quad (7)$$

The first equation shows that as the number of potential infringers increases, the optimal number of defensive patents required also increases. As explained earlier, this is consistent with the “strategic patenting” argument. The second equation, however, implies that as the overlap density of the technological space increases, the optimal size of the portfolio decreases. This is congruent with the “tragedy of anticommons” argument. More interestingly, the slope of the absolute overlap density effect is $(m + n_0)$ times larger than that of the fragmentation effect. Hence I expect that at higher levels of overlap density, the “tragedy of anticommons” effect overshadow the “fragmentation” effect. From an empirical perspective, we should see the overall impact of fragmentation and overlap density to be negative, especially in overcrowded technological areas.

This result actually sounds logical. Otherwise, we should have expected an ever-increasing patenting rate which is realistically impossible. In other words, the negative effect of the overlap density is the force which alleviates the exponential growth of the patenting rate and brings the system back into equilibrium; though into an inefficient one where each firm is more or less blocked with other firms’ patents and no further technological development can hypothetically take place.

3.3. The Mixture of ex ante and ex post Strategies

Now assume that the firm knows about the patents on which it might infringe before developing its technology and attempts to maximize its expected profit through negotiating licenses with l patent holders initially and defending itself against others based on the strength of its patent portfolio. To account for the patent royalties, I follow the same formulation used in Lemley and Shapiro (2007). Assume that the focal firm faces a linear demand curve: $Output = A + V - Price$, where parameter V reflects the value added to the firm's developed technology by the patents on which the firm might infringe and A represents the value of the product if none of these patents were used in the developed technology. Also assume that the firm's marginal cost of production, before accounting for any patent royalties, is C .

Define patentee i 's royalty rate r_i ($i = 1, \dots, l$). Furthermore, assume that each licensing negotiation will impose the fixed transaction cost of c_T leading to an aggregate transaction cost of $C_T = l \times c_T$. Given that the focal firm will secure a monopolistic market using its patents, it would set the price and the output level as follows:

$$price = \frac{1}{2}(A + V + R + C_T + C) \quad (8)$$

$$output = \frac{1}{2}(A + V - (R + C_T + C)) \quad (9)$$

Patentee i sets r to maximize $r \times output$ which yields a royalty rate of:

$$r_i = A + V - (R + C_T + C) \quad (10)$$

For the sake of simplification, assume that all the patents add the same value to the new technology. Therefore, in a symmetric equilibrium, we have $r_i = r$ and $R = l \times r$ leading to:

$$r = \frac{1}{l+1}(A + V - C - C_T) \quad (11)$$

And a stacked royalty rate of

$$R = \frac{l}{l+1}(A + V - C - C_T) \quad (12)$$

The corresponding profit level will be

$$\pi_l = \frac{1}{4} \left(\frac{2l+1}{l+1}(A + V) + \frac{1}{l+1}(C_T + C) \right) \left(\frac{1}{l+1}(A + V) - \frac{2l+1}{l+1}(C_T + C) \right) \quad (13)$$

Replacing this new profit level (π_l) with the one in the previous model (π_m) gives:

$$E(\pi) = \pi_l - (k + nd - l) \cdot c_i \cdot \frac{m}{n + n_0 + m} - n \cdot c_p \quad (14)$$

where $(k + nd - l)$ represents to maximum number of possible infringes and l can range from 0 to $(k + nd)$. Now maximizing the expected profit with respect to the number of licenses (l) and the number of new patents (n) yields

$$n^* = \sqrt{\frac{(k - l - d(m + n_0)) \cdot c_i \cdot m}{c_p}} - m - n_0 \quad (15)$$

It turns out that finding a general closed-form solution for l is not that straightforward. However, based on some loose assumptions, we can still derive the following results:

$$\frac{\partial n^*}{\partial k} > 0 \text{ ("Patent Proliferation" argument)} \quad (16)$$

$$\frac{\partial n^*}{\partial d} < 0 \text{ ("tragedy of anticommons" argument)} \quad (17)$$

$$\frac{\partial n^*}{\partial d} = -(m + n_0) \frac{\partial n^*}{\partial k} \quad (18)$$

(See the calculations in appendix A)

As it can be seen, the results do not change from what we found in the previous section. In other words, no matter how many of the potential infringes are taken care of through ex ante negotiations, fragmentation and overlap density still put their effects on patenting rate as long as there are some unresolved potential infringes out there. In addition to the aforementioned predictions, we can also show that:

$$\frac{\partial n^*}{\partial n_0} < 0 \quad (19)$$

It basically means that a firm with a stronger initial relevant portfolio will be less likely to engage in patent proliferation strategy. At extreme, when n_0 is very large, the new patents add little to the winning probability of the firm in potential disputes, yet add the fixed cost of filing and prosecution and the expected costs of new potential infringements. In conclusion, the main predictions of the model can be summarized as follows:

Prediction 1 (“Strategic Patenting”): firms will patent more aggressively when their new technologies draw on patents assigned to a disperse set of patent holders.

Prediction 2 (“Tragedy of Anticommons”): the extent to which outside right holders have overlapping claims has a negative impact on firms’ patenting rate.

Prediction 3: the stronger a firm’s initial relevant portfolio size, the less aggressively it will patent subsequently.

Prediction 4: the overall impact of fragmentation and overlap density on firms’ patenting rate will be negative.

4. Empirical Methodology

Our goal in this section is to empirically test the theoretical predictions outlined in the previous section by using an extensive dataset of firms' patenting behavior in the semiconductors industry. In the first step, I test the effect of each of the main independent variables, i.e. fragmentation, overlap density, and initial patent portfolio strength, on the firms' patenting rate. Similar to Hall and Ziedonis (2001) and Ziedonis (2004), I use a baseline estimate to identify levels of patenting beyond or below what is otherwise predicted. In the next step, I add the interaction of fragmentation and overlap density to investigate their joint effect. The following sections describe the data set, the main variables and the empirical model used in the estimations.

4.1. Data

I use a panel of patenting behavior of public firms in the semiconductors industry from 1980 to 1999. The semiconductors industry is known as a research intensive industry with a large degree of complementarity between technologies developed by different players in the market (Grindley and Teece, 1997; Cohen et al., 2000). Moreover, new technologies and products in the semiconductors industry are highly dependent on several prior inventions developed both internally and externally (Grindley and Teece, 1997; Shapiro, 2001). In fact, several previous studies have found evidence for the extensive use of patent proliferation strategies especially among capital-intensive firms in the industry (Hall and Ziedonis, 2001; Ziedonis, 2004). Furthermore, several incidences of "patent thicket" formation in the industry have previously reported (Grindley and Teece, 1997; Shapiro, 2001). All these characteristics make the semiconductors industry an appropriate benchmark to test our predictions.

In order to construct the dataset, a longitudinal sample of all publicly-traded U.S. firms reporting their main line of business to be in semiconductors (SIC3674) have been compiled from COMPUSTAT⁷. The resulting dataset is an unbalanced panel with 201 firms. 55 firms with less than three years of valid data were subsequently removed from the sample. To identify the patents assigned to the sampled firms, I used the NBER patent dataset along with the dynamic matching table to Compustat provided by NBER⁸. The sampled firms collectively were granted 28,395 U.S. patents between 1980 and 1999. However, in 683 observations a firm does not have any successful patent application in a given year, resulting in missing values for the main independent variables. I further remove these observations from the data. The final table contains 528 observations on 78 firms between 1980 and 1999.

4.2. Variables

In this section I describe the variables which are used in our empirical analysis.

Dependent Variable: The main outcome variable is the number of successful patent applications made by each firm in a given year as our dependent variable. The data is extracted from NBER patent dataset. The level of observation is firm year.

Independent Variables: In order to test the predictions of the theoretical model, I need to derive the measures of fragmentation, overlap density and initial portfolio strength.

In order to measure the fragmentation level, I use the fragmentation index proposed by Ziedonis (2004). A number of studies have used this measure previously especially in complex industries such as semiconductors (Ziedonis, 2004; Schankerman and Noel, 2006; Graevenitz et al., 2010).

⁷Prior to 1980, truncation bias in the constructed variables could potentially distort the results.

⁸ See Hall, Jaffe, and Trajtenberg (2001) for details.

Fragmentation index is developed based on the assumption that each citation to another firm's patents can potentially introduce an ex post infringement possibility during the commercialization phase. Thus as the number of different entities whose patents are cited by the focal firm increases, the number of ex post potential infringements that it faces also increases. Ziedonis (2004) suggest that it is less feasible for a firm with a high fragmentation index to engage in numerous ex ante negotiations with potential right holders to secure rights to their patented inventions before developing upon them further. The index is based on the Herfindahl index of citation concentration:

$$Fragmentation_{i,t} = 1 - \sum_{j=1}^J \left(\frac{NBCITES_{ij,t}}{NBCITES_i} \right)^2$$

where $NBCITES_{ij,t}$ refers to the number of backward patent citations to each unique entity j cited by firm i 's patents at year t . $NBCITES_i$ is the total number of backward patent citations listed on firm i 's patents at year t . Self-citations and citations to non-patent materials are excluded. As can be seen, the measure takes the value zero if all the citations of the firm are pointed to one other entity. As the dispersion of references increase, so does the fragmentation index. A firm with 10 citations all pointed to one entity has a fragmentation index of zero, whereas a firm with 10 citations each pointed to a different entity will end up with a fragmentation index of 0.9.

In order to derive a measure of overlap density, again I take advantage of the citation data. Given that each intra-entity patent citation represents some degree of technological overlap between the two entities, the total number of intra-entity citations in a firm's citation network divided by the maximum potential number of such citations can be used as a proxy for the crowdedness of the firm's external technology base.

Figures 1a and 1b give a better illustration of this phenomenon. Each directional link from one entity to another represents one or more patent citations from the former to one or more patents assigned to the latter. In both figures, firm i cites the same number of external entities. However, in figure 1a, there is no citation between the cited entities; whereas in figure 1b, there is a dense web of citations between them. Assuming that each citation between each pair of entities show some level of technological overlap between them, then we expect a much higher overlap density among the entities in figure 1b compared to those in figure 1a. In other words, any new patent filed in the technological space illustrated in figure 1b is much more likely to overlap with patents assigned to several other entities comparing to a new patent filed in the technological space depicted in figure 1a.

 Insert Figure 1 about here

I use a 5 year window to count the number of intra-entity citations in the citation network of a firm. I expect that as a citation grows older, its relevance to the current technological stock of the firm erodes. Considering the pace of technological development in the semiconductors industry, 5 years seems to be a reasonable time frame. I further divided the number of intra-entity citations in the citation network of a firm by the maximum potential number of such intra-entity citations to have a normalized measure of overlap density ranging from 0 to 1. If firm i cites n different entities in a given time window, then the maximum number of potential directional citations between them is $n \times (n - 1)$. Thus the overlap density index is constructed as follows:

$$overlap_{i,t} = \frac{\text{total number of intraentity links between } t - 5 \text{ and } t}{n(n - 1)}$$

Note that all the citations from firm A’s patents to firm B’s patents will be counted as one directional link from firm A to firm B.

Finally, I use the citation data once again to construct a measure for the strength of the firm’s initial patent portfolio. Note that while a simple count of the firm’s patents in the previous periods may give us some rough measure of initial portfolio strength, such measure does not tell us directly the degree to which these patents can be used in future disputes or negotiations.

Consider two firms, both having 100 patents in their portfolio. However, assume that the patents assigned to one of them are only cited by one other firm, whereas the patents assigned to the other are cited by numerous other firms. I basically expect that the patent portfolio of the latter is much stronger than that of the former. Therefore, I expect the latter to be less dependent on a patent proliferation strategy to protect its new technology.

To implement this feature, I construct a similar measure to fragmentation index while I replace the number of backward citations with the number of citations received from other entities in a specific prior time frame. Using a Herfindahl index of received citations concentration in the 5 year window prior to the current year, I construct the “portfolio strength index” as follows:

$$Portfolio\ Strength_{i,t} = 1 - \sum_{j=1}^J \left(\frac{NBCITES_{ji,[t-1,t-5]}}{NBCITES_{i,[t-1,t-5]}} \right)^2$$

where $NBCITES_{ji,[t-1,t-5]}$ refers to the number of patent citations to firm i ’s patents listed in patents assigned to entity j during the 5 years prior to t . $NBCITES_{i,[t-1,t-5]}$ represents the total number of citations to firm i ’s patents listed in patents assigned to all J entities during the 5 years prior to year t . A value close to zero shows that the patents assigned to i are cited by few firms;

while a value close to one shows that its patents are cited frequently by several entities in the previous 5 years.

Control Variables: To identify the firms' patenting rate beyond or below what is otherwise predicted, similar to Ziedonis (2004), I construct a baseline estimate using the key determinants of patenting rate identified in the previous research (Pakes and Griliches, 1980; Kortum and Lerner, 2000; Hall and Ziedonis, 2001). The variables used in the baseline estimate will be used as controls in all the other estimations. The key control variables in the baseline model are as follows.

R&D Intensity: the logarithm of the amount of R&D spending (in \$M 1984) during the year in which the patent applications were filed divided by the number of employees in the same year.

Firm's size: the logarithm of the number of employees.

Capital-intensity: the logarithm of the deflated value of the firm's total assets (in \$M 1984) divided by the number of employees.

Time Dummies: a dummy variable for each year from 1980 to 2000 to control for the microeconomic trends such as periods of technological recession or ferment.

Table 1 shows the summary statistics for the main variables used in the estimations to follow. As it can be seen, an average firm in the data makes around 40 successful patents each year; spends \$20.95 million (1984) dollars per 1000 employees on R&D. It has a fragmentation index of 0.88, an overlap density of 0.48, and initial portfolio strength of 0.86. The stats are quite consistent with the ones reported in Ziedonis (2004).

Insert Table 1 about here

4.3. Model Specification

The dependent variable, number of successful patent applications of each firm in a given year, is a count variable. Hence, following previous literature on patenting and R&D (e.g. Hausman, Hall, and Griliches, 1984), I use a negative binomial estimation method with firm fixed effects and “Robust” standard errors. Previous studies (Cameron and Trivedi, 1998) show that negative binomial estimation is a better alternative to a pure Poisson model when the sample variance exceeds the sample mean. However, in the robustness checks section I also report the results of the Poisson estimation with firm fixed effects and “Robust” standard errors. I further re-estimate the models with conditional quasi-maximum likelihood (QML) method based on the fixed-effect Poisson model developed by Hausman, Hall, and Griliches (1984). Firm fixed effects basically control for any time-invariant characteristic of firms. Some firms might be simply better at assimilating and using external knowledge and develop new technologies based on them. Using firm fixed effects, I can control for unobserved time-invariant heterogeneity among firms.

5. Results

Table 2 shows the results derived from the negative binomial estimation with firm fixed effects and “Robust” standard errors. Column 1 presents baseline estimates including all control variables and year dummies. Consistent with previous findings, I find that larger firms and more capital intensive firms have a higher propensity to patent. The insignificance of the R&D

intensity is likely because firms' R&D intensity is relatively stable over time and thus its effect is essentially captured by firm fixed effects in the model.

The results for our main three independent variables are presented in column 2. Results show strong support for the first three predictions. Fragmentation index has a strong positive effect on the patenting rate. In fact, the results show that the average patenting rate of firms with high fragmentation index is more than 5 times larger than that of the firms with low fragmentation index. Both the size and the sign of the fragmentation index coefficient are consistent with the results reported in Ziedonis (2004). The effect of overlap density is also in the predicted direction and significant. A 10% increase in the overlap density leads to a more than 80% decrease in patenting rate. Furthermore, as expected, stronger initial portfolio strength decreases the patenting rate of the firm, although the effect is smaller in size relative to the effect of overlap density.

In column 3, I add the interaction of fragmentation and overlap density. Also, to make sure that the results in the previous model are not driven by the size of the firms' citation stock which appears in the denominator of both fragmentation and overlap density indexes, I add the reverse of firm's total citation count as a control variable to the new model. The results confirm the last prediction and show that only firms with low overlap density have higher patenting rates. On the other hand, firms with high fragmentation index and high overlap density have lower patenting rates relative to the baseline. It is in line with the idea that in dense and crowded technological areas, firms with higher fragmentation index would prefer to use other mechanisms such as secrecy or joint ventures to protect their new technologies. Fragmentation index and initial portfolio strength remain negative and significant. However the coefficient on overlap density turns to positive once I add the interaction term. The positive coefficient is consistent with the

idea that, in technological spaces with high overlap density, firms with low fragmentation index may engage in an aggressive patenting behavior for reasons other than protecting their technologies, such as blocking competitors.

Insert Table 2 about here

Table 3 shows the results for several robustness checks. The first column shows the full model estimated using Poisson method with fixed effects and “Robust” standard errors. The results are essentially the same. Columns 2 and 3 show the results for negative binomial estimation with random effects and conditional quasi-maximum likelihood (QML) method based on the fixed-effect Poisson respectively. As it can be seen, the results remain mainly unchanged. In column 4, I replace the fragmentation index with a three-year moving-average of it to further control for the general characteristics of a firm’s technological base over time rather than its sudden technological shifts year to year. However, it does not affect the results.

Insert Table 3 about here

6. Discussion and Conclusion

The cumulative nature of innovation and the blurry boundaries of patents have created a dense web of overlapping property rights with numerous right holders in different technological areas.

The “strategic patenting” argument suggests that firms facing a more fragmented external technology market will engage in patent proliferation strategies to increase their bargaining

power in potential ex-post disputes or negotiations. On the other hand, the “tragedy of anticommons” proposes that a fragmented technological market with numerous right holders claiming overlapping regions of the technological space harms further technological development due to severe costs of ex-ante negotiations and ex-post infringement disputes, leading to a decline in firms’ patenting rate.

To see how these two phenomena interact with each other, I developed a model which captures the assumptions underlying both phenomena: more relevant patents in a firm’s patent portfolio increases its bargaining power in future disputes; and new patents in crowded technological spaces increase the chance of infringing other patents and introduce potential infringement costs. The theoretical model suggests that a higher technological fragmentation can indeed result in a higher patenting rate. Furthermore, overlap density, i.e. the extent to which firms claim overlapping regions of the technological space, negatively affects the patenting rate of firms. More importantly, the model predicts that, in fact, overlap density has a relatively stronger impact on patenting rate.

I investigated the model predictions in the semiconductor industry. The empirical analysis supports both “strategic patenting” and “tragedy of anticommons” arguments. In addition, it shows that overlap density does indeed have a stronger impact on patenting rate. Furthermore, consistent with the model predictions, I found that firms with higher initial portfolio strength are less likely to engage in patent amassing strategies.

This study makes several contributions to the literature. This paper represents the first work which studies both “strategic patenting” and “tragedy of anticommons” arguments and provides empirical support for both of them. It shows that both phenomena can co-exist in a single

industry. It further suggests, theoretically and empirically, that the overlap density, i.e. the “tragedy of anticommons”, has a relatively stronger impact on the patenting behavior of firms. My findings also speak to a range of policy implications. At the broad level, they show that the side effects of the current patenting system may overshadow its key purpose: to protect inventors from imitation and create incentives for them to incur the heavy costs of innovation. An overcrowded, fragmented technological area may indeed discourage further technological development. At the firm level, they propose that defensive strategic patenting cannot be a stable strategy in the long term, considering that it leads to more technological fragmentation and higher overlap density. In fact, several previous studies suggest that pursuing defensive strategic patenting may eventually lead to technological stagnation in different industries (Shapiro, 2001; Ziedonis, 2004).

However, this study also has limitations. First, the model developed in the paper focuses on defensive nature of patenting. As explained earlier, firms may patent for several other reasons. For instance, they may patent to block their competitors (Cohen et al., 2000; Parchomovsky and Wagner, 2004) or to obtain financing (Hall et al., 2005). Patenting for the purpose of blocking competitors from patenting in relevant technology areas in particular seems to be a common strategy in biotechnology industry. Future research could include the effect of such strategies on firms’ patenting behavior. Moreover, the empirical analysis is solely based on the data from a single industry, semiconductors. While the theoretical model does not rely on any strong assumptions regarding the characteristics of the underlying industry or technology, further empirical analysis in other sectors and industries is required to test the generalizability of the predictions. Finally, considering that firms’ patenting behavior affects the overlap density and

the technological fragmentation level, a temporal analysis of both phenomena and their interactions seems to be a promising avenue for future research.

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Figure 1: Overlap Density based on Patent Citations

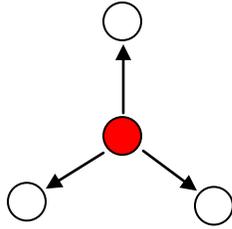


Figure 1a: a technological space with overlap density index equals zero.

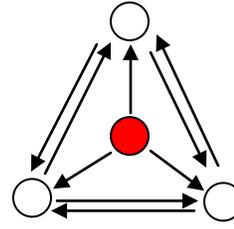


Figure 2a: a technological space with overlap density index equals one.

A → B: a citation from a patent assigned to A to a patent assigned to B

● : the focal firm

○ : a firm in the citation network

Table 1: Sample Statistics

	Mean	SD	Median	Min	Max
Patent Count	39.68	152.65	5	1	1552
Fragmentation	0.88	0.17	0.94	0	0.99
Overlap Density	0.48	0.22	0.47	0.02	1
Portfolio Strength	0.86	0.20	0.93	0	0.98
Capital Intensity (\$M 1984/1000 employ)	162.14	133.91	131.49	12.94	1539.67
R&D Intensity (\$M 1984/1000 employ)	20.95	18.79	15.68	0	127.53
Firm Size (1000 employ)	4.41	11.86	0.56	0.001	89.87

Table 2: Main Results: Patenting Behavior of Semiconductor Firms between 1980-1999

Negative Binomial Regression with Firm Fixed Effects and Robust Standard Errors			
Dependent Variable: Patent Count			
	baseline estimation (1)	Adding Main Variables (2)	Adding Interaction Term (3)
Fragmentation * Overlap Density			-11.87** (1.933)
Fragmentation		1.68* (0.700)	9.03** (1.931)
Overlap Density		-2.11** (0.271)	8.76** (1.827)
Portfolio Strength		-0.38+ (0.225)	-0.59** (0.217)
Ln(Capital Intensity)	0.31** (0.100)	0.28** (0.091)	0.27** (0.089)
Ln(R&D Intensity)	-0.08 (0.083)	0.07 (0.084)	0.07 (0.082)
Ln(Firm Size)	0.18** (0.060)	0.14+ (0.077)	0.16* (0.079)
Reverse of Total Citation Count			-1.19 (1.41)
Constant	0.13 (0.483)	-0.23 (0.807)	-6.73** (1.877)
Year Dummies	Yes	Yes	Yes
Observations	522	522	522
Number of unique firms	78	78	78
Log-likelihood	-1473	-1423	-1405
Chi-squared (p-value)	197.5 (0.000)	363.0 (0.000)	509.2 (0.000)

** p<0.01, * p<0.05, + p<0.1

Method of estimation is panel negative binomial with conditional firm fixed effects.

“Robust” standard errors are shown in parentheses.

Table 3: Robustness Checks: Patenting Behavior of Semiconductor Firms between 1980-1999

Dependent Variable: Patent Count				
	Poisson Regression with Firm Fixed Effect (1)	Negative Binomial with Random Effects (2)	QML Fixed Effect Poisson Model (3)	Replacing Frag with 3 yr m.a. (4)
Fragmentation * Overlap Density	-18.04** (3.608)	-12.94** (1.768)	-18.04** (3.608)	-19.30** (3.201)
Fragmentation	9.23* (3.866)	9.266** (1.839)	9.23* (3.866)	10.43** (2.306)
Overlap Density	13.03** (3.654)	9.52** (1.671)	13.03** (3.654)	15.03** (2.991)
Portfolio Strength	-1.07** (0.266)	-0.50** (0.181)	-1.07** (0.266)	-0.57* (0.275)
Ln(Capital Intensity)	0.43* (0.189)	0.35** (0.077)	0.43* (0.189)	0.39** (0.110)
Ln(R&D Intensity)	0.25 (0.186)	0.10 (0.065)	0.25 (0.186)	-0.07 (0.094)
Ln(Firm Size)	0.64** (0.172)	0.298** (0.050)	0.64** (0.172)	0.24* (0.094)
Reverse of Total Citation Count	-7.06* (3.124)	-1.50 (1.329)	-7.06** (1.060)	-3.01** (1.155)
Constant	-	-7.42** (1.777)	-	-7.97** (2.117)
Year Dummies	Yes	Yes	Yes	Yes
Observations	522	542	522	381
Number of unique firms	78	98	78	50
Log-likelihood	-1921	-1849	-1921	-1129
Chi-squared (p-value)	30282 (0.000)	809.3 (0.000)	16413 (0.000)	685.26 (0.000)

** p<0.01, * p<0.05, + p<0.1

Method of estimation for each model is shown on top of the column.

“Robust” standard errors are shown in parentheses.

Appendix A: Calculations for the Model with ex ante Licensing Negotiations

I have already shown that:

$$E(\pi) = \pi_l - (k + nd - l) \cdot c_i \cdot \frac{m}{n + n_0 + m} - n \cdot c_p$$

where

$$\pi_l = \frac{1}{4} \left(\frac{2l+1}{l+1} (A+V) + \frac{1}{l+1} (C_T + C) \right) \left(\frac{1}{l+1} (A+V) - \frac{2l+1}{l+1} (C_T + C) \right)$$

Solving the first order conditions for n yields:

$$n^* = \sqrt{\frac{(k - l - d(m + n_0)) \cdot c_i \cdot m}{c_p}} - m - n_0$$

The first order conditions for l gives the following equation:

$$\frac{-c_i \cdot m}{n + n_0 + m} - (A + V)C + \frac{1}{4(l^* + 1)^2} ((A + V)^2 - (C + C_T)^2) - \frac{2l^* + 1}{2(l^* + 1)} c_T (C + C_T) = 0$$

Replace n with n^* from the previous step. Now by using the chain rule we can calculate the first derivative of l^* with respect to d :

$$\left[\frac{\sqrt{c_p \cdot c_i \cdot m}}{2(k - l - d(m + n_0))\sqrt{k - l - d(m + n_0)}} + \frac{(A + V)^2 - (C + C_T)^2 + (l + 1)(C + C_T)}{2(l + 1)^2} \right] \frac{\partial l^*}{\partial d} = -(n_0 + m)$$

Considering that $(A + V)^2 - (C + C_T)^2$ is positive, one can conclude that

$$\frac{\partial l^*}{\partial d} < 0$$

Following the same line of reasoning, we can see that:

$$\left[\frac{\sqrt{c_p \cdot c_i \cdot m}}{2(k-l-d(m+n_0))\sqrt{k-l-d(m+n_0)}} + \frac{(A+V)^2 - (C+C_T)^2 + (l+1)(C+C_T)}{2(l+1)^2} \right] \frac{\partial l^*}{\partial k} = 1$$

which yields:

$$\frac{\partial l^*}{\partial k} < 0$$

and,

$$\frac{\partial l^*}{\partial d} = -(n_0 + m) \frac{\partial l^*}{\partial k}$$

Now by taking the derivative of n^* with respect to k and l , we have:

$$\frac{\partial n^*}{\partial k} = \frac{1 - \frac{\partial l^*}{\partial k}}{2} \sqrt{\frac{c_p}{c_i \cdot m}} \frac{1}{\sqrt{k-l-d(m+n_0)}}$$

$$\frac{\partial n^*}{\partial d} = \frac{-(m+n_0) - \frac{\partial l^*}{\partial d}}{2} \sqrt{\frac{c_p}{c_i \cdot m}} \frac{1}{\sqrt{k-l-d(m+n_0)}}$$

Assuming that $\frac{\partial l^*}{\partial d}$ is smaller than 1 (which is fairly plausible for large values of A and c_i) we

finally have:

$$\frac{\partial n^*}{\partial k} > 0$$

$$\frac{\partial n^*}{\partial d} < 0$$

$$\frac{\partial n^*}{\partial d} = -(m+n_0) \frac{\partial n^*}{\partial k}$$

Using similar calculations we can also show that

$$\frac{\partial n^*}{\partial n_0} < 0$$